AD-781 386

ENERGY SPECTRA OF THE OCEAN SURFACE: III. MODULATION BY A SURFACE CURRENT

Kenneth M. Case, et al

Physical Dynamics, Incorporated

Prepared for:

Rome Air Development Center
Defense Advanced Research Agency

December 1973

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER RADC-TR-74-110	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER AD-781386
4. TITLE (and Subtitle) ENERGY SPECTRA OF THE OCEAN SURFACE: III MODULATION BY A SURFACE CURRENT		5. TYPE OF REPORT & PERIOD COVERED Semi-Annual (Jul - Dec 73)
		6. PERFORMING ORG. REPORT NUMBER PD-73-047
7. AUTHOR(s) Kenneth M. Case Kenneth M. Watson Bruce J. West		8. CONTRACT OR GRANT NUMBER(s) F30602-72-C-0494
9. PERFORMING ORGANIZATION NAME AND ADDRESS Physical Dynamics, Inc. P.O. Box 1069 Berkeley, Ca. 94701		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62301E/16490402
11. CONTROLLING OFFICE NAME AND ADDRESS Defense Advanced Research Projects Agency ATTN: STO 1400 Wilson Blvd., Arlington, Va. 22209		12. REPORT DATE December 1973
		13. NUMBER OF PAGES 45
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) RADC/OCSE ATTN: L. Strauss GAFB, N.Y. 13441		15. SECURITY CLASS. (of this report) UNCL
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Same

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Eigenmodes
WKB approximation
Surface Currents
Wave Spectrum

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Several models for the analysis of the modulation of the ocean surface wave spectrum by a prescribed surface current are reviewed. Both an eigenmode description and WKB approximation will be studied in two horizontal dimensions. The results will be expressed as a modulation of the surface wave amplitude power spectrum.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ENERGY SPECTRA OF THE OCEAN SURFACE:
III. MODULATION BY A SURFACE CURRENT

Kenneth M. Case
Kenneth M. Watson
Bruce J. West

Contractor: Physical Dynamics, Incorporated
Contract Number: F30602-72-C-0494
Effective Date of Contract: 1 May 1972
Contract Expiration Date: 31 December 1974
Amount of Contract: $91,818.00
Program Code Number: 2E20

Principal Investigator: J. A. Thomson
Phone: 415 848-3063

Project Engineer: J. J. Simons
Phone: 315 330-3055

Contract Engineer: L. Strauss
Phone: 315 330-3055

This research was supported by the Defense Advanced Research Projects Agency of the Department of Defense and was monitored by Leonard Strauss RADC (OCSE), GAFB, NY 13441 under Contract F30602-72-C-0494.

PUBLICATION REVIEW

This technical report has been reviewed and is approved.

Leonard Strauss
RADC Project Engineer

ABSTRACT

Several models for the analysis of the modulation of the ocean surface wave spectrum by a prescribed surface current are reviewed. Both an eigenmode description and WKB approximation will be studied in two horizontal dimensions. The results will be expressed as a modulation of the surface wave amplitude power spectrum.

TABLE OF CONTENTS

	Page
ABSTRACT	iii
1.0 INTRODUCTION	1
2.0 DESCRIPTION OF THE PROBLEM	2
3.0 ANALYSIS OF THE COUPLING	7
4.0 THE INITIAL VALUE CONDITION	20
5.0 THE WKB APPROXIMATION	32
REFERENCES	37
FIGURE	38

1.0 Introduction

In this paper we review several models for analysis of the modulation of the ocean surface wave spectrum by a prescribed surface current. A quasi-linear approximation will be used. The free surface waves will be treated in the linear approximation and the coupling to the surface current will be proportional to the product of the surface wave and surface current amplitudes. Both a modal description and the WKB approximation will be studied. The results will be expressed as a modulation of the surface wave amplitude power spectrum.

The notation of Parts I[1] and II[2] will generally be followed.

2.0 Description of the Problem

The model system we consider in this paper is that of an irrotational, incompressible ocean. We treat the surface in two horizontal dimensions with the undisturbed ocean surface chosen to be the z=0 plane of a rectangular coordinate system with z-axis directed upward. The depth of the water is assumed to be very large compared with all wavelengths of interest. The assumption of irrotationality allows us to use a velocity potential description of the surface waves $\left[\phi(\vec{r},t)\right]$ and the vertical displacement of the water surface from equilibrium is written as $\zeta(\vec{r},t)$. The prescribed surface current is assumed to be parallel to the x-axis and a superposition of modes of the form

$$\underset{\sim}{U}(\vec{r},t) = \hat{i} \sum_{K} U_{K} \cos(K\xi) \qquad (2.1)$$

where

$$\xi = x - c_{I}t \qquad (2.2)$$

with c_I a phase velocity, K representing a set of wave numbers and the U_K a set of amplitudes.

The linearized equations to determine ϕ and ζ were obtained for one-dimensional waves by Zachariasen[(3)] and Milder[(4)] and also in Part II. These can be generalized to

two dimensions; the resulting equations are[5]

$$\frac{\partial\phi}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\,\phi + g\,\zeta = 0,$$

$$\frac{\partial\zeta}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\,\zeta + \zeta\,\nabla_s\cdot\underset{\sim}{U} = \text{\textcircled{\#}}_s\,\phi \quad . \tag{2.3}$$

Here g is the acceleration of gravity, ϕ is the velocity potential at the surface, and ζ is the vertical displacment of the surface from the plane $z=0$. The operator ∇_s is the gradient in the (x,y) plane and

$$\text{\textcircled{\#}}_s = \sqrt{-\nabla_s^2} \quad .$$

Following the notation of Part I, we write

$$\phi(\vec{r},t) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\, e^{kz}\, \frac{v_k}{\sqrt{2}\,k}\left(b_{\vec{k}}^{(+)} + b_{-\vec{k}}^{(+)*}\right) \tag{2.4}$$

and for the surface displacement from equilibrium

$$\zeta(\vec{r},t) = \sum_{\vec{k}} ie^{i\vec{k}\cdot\vec{r}}\, \frac{1}{\sqrt{2}\,k}\left(b_{\vec{k}}^{(+)} - b_{-\vec{k}}^{(+)*}\right) \tag{2.5}$$

where $v_k = \sqrt{g/k}$, the $b_{\vec{k}}^{(+)}$ are expansion amplitudes, and the sum runs over surface wave number vectors $\vec{k}$. Substitution of Eqs. (2.4) and (2.5) into (2.3) leads to the equation for the $b_{\vec{k}}^{(+)}$:

$$i\,\dot{b}^{(+)}_{\vec{k}} - \omega_k\, b^{(+)}_{\vec{k}} = \sum_{K} \frac{kU_K}{4}\left\{\frac{k_x-K}{|\vec{k}-\vec{K}|}\, e^{-i\Omega_K t}\left[\frac{k_x}{k_x-K} + \frac{\omega_k}{\omega_{|\vec{k}-\vec{K}|}}\right] b^{(+)}_{\vec{k}-\vec{K}} + \frac{k_x+K}{|\vec{k}+\vec{K}|}\, e^{i\Omega_K t}\left[\frac{k_x}{k_x+K} + \frac{\omega_k}{\omega_{|\vec{k}+\vec{K}|}}\right] b^{(+)}_{\vec{k}+\vec{K}}\right\} \tag{2.6}$$

where we have seen in Part II that the coupling between $b^{(+)}_{\vec{k}}$ and $b^{(+)*}_{-\vec{k}}$ $\left(=b^{(-)}_{\vec{k}}\right)$ is very weak and therefore ignored and

$$\Omega_K \equiv Kc_I \text{ and } \omega_k \equiv \sqrt{g|\vec{k}|} \quad . \tag{2.7}$$

Continuing to follow the notation of Part I, we set the eigenmode expansion amplitudes to be of the form

$$b^{(+)}_{\vec{k}} \equiv e^{-i\omega_k t}\, q_{\vec{k}}\, 2^{-1/2} \tag{2.8}$$

and define the complex amplitude

$$Z(\vec{r},t) \equiv \sum_{\vec{k}} e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\, q_{\vec{k}}\, k^{-1} \quad . \tag{2.9}$$

In terms of this complex amplitude we may write the surface displacement as,

$$\zeta(\vec{r},t) = -\,\mathrm{Im}\left\{Z(\vec{r},t)\right\} = \frac{i}{2}\left(Z(\vec{r},t) - Z^*(\vec{r},t)\right) \quad . \tag{2.10}$$

Until now our discussion has been concerned with the mechanical interaction of waves and the resulting surface displacement. Suppose we shift our focus and determine the spectral development induced by the interaction of the surface current and surface gravity waves. Let us suppose that at a given time t_o we observe $q_{\vec{k}}$ to have the value $Q_{\vec{k}}$. A series of such observations at time intervals long compared to correlation times will lead to an ensemble of values for the $q_{\vec{k}}$'s. We shall assume the different $Q_{\vec{k}}$'s to be uncorrelated so that for an ensemble average denoted by the brackets $\langle \ldots \rangle$ we have the relations,

$$\langle Q_{\vec{k}} \rangle = \langle Q_{\vec{k}} Q_{\vec{\ell}} \rangle = 0 \quad ,$$

$$\langle Q_{\vec{k}} Q^*_{\vec{\ell}} \rangle = \langle |Q_{\vec{k}}|^2 \rangle \delta_{\vec{k}\vec{\ell}} \quad . \tag{2.11}$$

We can use these considerations to construct the correlation function for the surface displacement ζ between two points separated by a distance $\vec{x}$ as follows,

$$C(\vec{x}) \equiv \langle \zeta(\vec{r},t)\, \zeta(\vec{r}+\vec{x},t) \rangle$$

$$= \sum_{\vec{k}} \tfrac{1}{2} \left\{ P(\vec{k}) + P(-\vec{k}) \right\} e^{i\vec{k}\cdot\vec{x}} \tag{2.12}$$

where $P(\vec{k})$ is the power spectrum of the complex amplitude Z. Using the expression for the surface displacement in terms of the complex amplitude, i.e., Eq. (2.10), in Eq. (2.12) yields the expression,

$$P(\vec{k}) = \frac{1}{2(2\pi)^2} \int d^2x \; e^{-i\vec{k}\cdot\vec{x}} \langle Z^*(\vec{r},t)\; Z(\vec{r}+\vec{x},t)\rangle$$

$$= \langle |Q_{\vec{k}}|^2\rangle / 2k^2 \;, \tag{2.13}$$

which is the power spectrum in terms of the measured slope variables.

The corresponding power spectrum of the surface displacement ζ is defined by Phillips[(6)] as

$$\Psi(\vec{k}) = \tfrac{1}{2}\left\{P(\vec{k}) + P(-\vec{k})\right\} \;. \tag{2.14}$$

The normalization of the functions $\Psi(\vec{k})$ and $P(\vec{k})$ are such that

$$\sum_{\vec{k}} \Psi(\vec{k}) = \sum_{\vec{k}} P(\vec{k}) = \langle \zeta^2 \rangle \;. \tag{2.15}$$

3.0 Analysis of the Coupling

In the remainder of this paper will assume that there is a single nondispersive internal wave which gives rise to the surface current. We write Eq. (2.1) as

$$\vec{U} = \hat{i}\, U_o \cos K\xi \tag{3.1}$$

where the wavelength of the internal wave ($2\pi/K$), i.e., surface current, is very much larger than the surface waves of interest. Thus, we have the relation

$$k \gg K \ , \quad \text{all } k \text{ of interest}. \tag{3.2}$$

To simplify our two-dimensional problem we parameterize the dependence of the surface wavenumber in the direction orthogonal to the current by introducing

$$p \equiv \hat{j}\cdot\vec{k} \tag{3.3}$$

into Eq. (2.6). With p as a fixed parameter in Eq. (2.6), the component of $\vec{k}$ parallel to the current $\vec{U}$ is written as,

$$k_x = nK \ , \quad n = 1, 2, \ldots. \tag{3. 4}$$

Eq. (3.4) is reasonable because of condition (3.1); i.e., we can establish the periodic boundary conditions for the surface gravity waves over an arbitrary patch of ocean.

Implementing Eq. (3.4) allows us to replace the eigen-mode amplitudes $b_{\vec{k}}^{(+)}$ by a discrete set of quantities B(n):

$$b_{\vec{k}}^{(+)} \equiv e^{-in\Omega t}\, B(n) \quad , \tag{3.5}$$

where

$$\Omega \equiv Kc_I \tag{3.6}$$

is the frequency of an internal wave of wavenumber k and phase velocity c_I. Substituting the discrete variables defined by Eq. (3.5) into the interaction equation [Eq. (2.6)] results in

$$i\,\dot{B}(n) - E_n\, B(n) = f(n)\, B(n+1) + g(n)\, B(n-1) \tag{3.7}$$

where

$$g(n) = n\,\frac{KU_o}{4}\,\frac{n-1}{\sqrt{(n-1)^2+(p/K)^2}}\left\{\frac{n}{n-1}+\frac{\omega_n}{\omega_{n-1}}\right\} , \tag{3.8}$$

$$f(n) = n\,\frac{KU_o}{4}\,\frac{n+1}{\sqrt{(n+1)^2+(p/K)^2}}\left\{\frac{n}{n+1}+\frac{\omega_n}{\omega_{n+1}}\right\} \tag{3.9}$$

and

$$E_n \equiv \omega_n - n\Omega \tag{3.10}$$

with $\omega_n \equiv \omega_{|\vec{k}|}$. A further transformation on Eq. (3.7) puts it into the form,

$$i \frac{\partial \psi}{\partial t} = H\psi$$

where H is Hermitian. Indeed, the transformation

$$B(n) = \gamma_n \psi(n) \tag{3.11}$$

leads to the equation, using Eq. (3.7),

$$i \dot{\psi}(n) - E_n \psi(n) = V_{n,n+1} \psi(n+1) + V_{n,n-1} \psi(n-1) \tag{3.12}$$

where the matrix elements are given by

$$V_{n,n+1} = f(n) \gamma_{n+1}/\gamma_n$$

and (3.13)

$$V_{n,n-1} = g(n) \gamma_{n-1}/\gamma_n \quad .$$

The matrix $\underset{\sim}{V}$ will be symmetric if we choose

$$\gamma_{n-1}/\gamma_n = \left|f(n-1)/g(n)\right|^{\frac{1}{2}} \tag{3.14}$$

so that

$$V_{n,n+1} = \sqrt{f(n)\ g(n+1)}$$

and (3.15)

$$V_{n,n-1} = \sqrt{f(n-1)\ g(n)} \quad .$$

Eq. (3.12) now has the form of a "Schrödinger equation",

$$i\,\frac{\partial\psi}{\partial t} = (K + V)\psi \tag{3.16}$$

where

$$K_{nn'} = E_n\ \delta_{nn'} \ .$$

Eigensolutions to Eq. (3.16) are

$$\psi(n) = e^{-iE^\lambda t}\ \psi_\lambda(n) \quad , \tag{3.17}$$

where λ labels the eigenfunctions ψ_λ and eigenfrequencies E^λ. Substitution of (3.17) into (3.16) leads to the equation

$$(E^\lambda - E_n)\ \psi_\lambda(n) = V_{n,n+1}\ \psi_\lambda(n+1) + V_{n,n-1}\ \psi_\lambda(n-1) \ . \tag{3.18}$$

Because of physical limitations on our model, as well as mathematical limitations on our equations, it is useful to

truncate the set of Eq. (3.18) at some maximum value of n, say n_{max}. We suppose that n_{max} is chosen sufficiently large that for time intervals of interest this truncation does not affect our conclusions. Convenient boundary conditions are to be imposed on the ψ_λ , such as

$$\psi_\lambda(n_{min}) = \psi_\lambda(n_{max}) = 0 \ , \tag{3.19}$$

where n_{min} is an assumed minimum value of n.

The eigenfunctions $\psi_\lambda(n)$ are supposed to be so chosen that they satisfy the ortho-normality relations

$$\begin{aligned} \sum_n \psi_\lambda(n)\, \psi_{\lambda}'(n) &= \delta_{\lambda,\lambda'} \ , \\ \sum_\lambda \psi_\lambda(n)\, \psi_\lambda(n') &= \delta_{n,n'} \ . \end{aligned} \tag{3.20}$$

To discuss the solutions of Eq. (3.16) several parameter regimes must be recognized. There is, first, the condition of "resonance", defined by the relation

$$\frac{\partial E_n}{\partial n} = K\left\{\frac{k_x}{k}\, c_g - c_I\right\} = 0 \ , \tag{3.21}$$

where $c_g = (g/k)^{\frac{1}{2}}/2$ is the group velocity of the surface waves. The condition (3.21) determines a value of k_x , say k_x = NK, at which resonance occurs. Condition (3.2) implies that

$N >> 1$, so we may conveniently take N to be an integer. Eq. (3.21) can be rewritten in the more convenient form

$$c_I = c_g \cos\theta \tag{3.22}$$

where

$$\cos\theta = NK/\left[p^2 + N^2K^2\right]^{\frac{1}{2}} \quad ; \tag{3.23}$$

that is, θ is the angle between the surface current and $\vec{k}$. For values of the surface wavenumbers near resonance, i.e., $n \approx N$, we may express the diagonal matrix element in Eq. (3.16) by the expansion about $n = N$

$$E_n = E_N + (n-N) \left.\frac{\partial E_n}{\partial n}\right|_{n=N} + \frac{(n-N)^2}{2!} \left.\frac{\partial^2 E_n}{\partial n^2}\right|_{n=N} + \ldots .$$

Using Eq. (3.22), we may re-write this as

$$E_n \approx E_N - \alpha(n-N)^2 + \ldots \tag{3.24a}$$

where α is

$$\alpha = \frac{Kc_I}{4N}\left[3 \cos^2\theta - 2\right] \quad . \tag{3.24b}$$

From the form of α above, we anticipate that anomalous resonance effects will occur for

$$\cos^2\theta \approx \frac{2}{3}$$

or

$$\theta = 35^\circ . \tag{3.25}$$

We shall see that for $\theta < 35^\circ$ modes with n near N tend to be "trapped" in this neighborhood, whereas when $\theta > 35^\circ$, modes originally near N tend to spread indefinitely away.

When we are interested in mode numbers n near n = M, we can use Eqs.(3.14) and (3.15) to write

$$V_{n,n\pm1} = V_{n\pm1,n} = \left(\frac{U_o K}{2}\right) \sqrt{n(n\pm1)} \left[1 + \mathcal{O}\left(\frac{1}{M^2}\right)\right] , \tag{3.26}$$

and

$$\gamma_n = (M/n)^{\frac{1}{2}} \exp\left[(n-M) \left(\frac{5}{2}\cos^2\theta+1\right)/(2M)\right]\left[1 + \mathcal{O}\left(\frac{1}{M^2}\right)\right] , \tag{3.27}$$

where we have specified that $\gamma_M = 1$.

When the coupling V is sufficiently weak, we have the first order perturbation solution to Eq. (3.18):

$$\psi_\lambda(n) = \delta_{\lambda n} + \frac{V_{n,\lambda}}{E_\lambda - E_n}\left[\delta_{\lambda,n+1} + \delta_{\lambda,n-1}\right] ,$$

$$E^\lambda = E_\lambda . \tag{3.28}$$

We anticipate that this will be a useful approximate solution when

$$S_\lambda \equiv \left| \frac{V_{\lambda,\lambda+1}}{E_{\lambda+1}-E_\lambda} \right| << 1 . \tag{3.29}$$

This condition is most stringent at $\lambda = N$, where we introduce the parameter

$$S \equiv \left| \frac{V_{N,N+1}}{E_{N+1}-E_N} \right| \cong \left| 2U_o \, N^2 / \left[c_I (3 \cos^2\theta - 2) \right] \right| \quad . \tag{3.30}$$

When S and S_λ (all λ) are sufficiently small, we can use perturbation theory. When S_λ is not small and when λ is not near a resonant mode, the WKB method is probably the simplest to discuss the surface wave modulation. The WKB method fails near resonance, so for S_λ, large special techniques are required.

For convenient reference, in Fig. (1) we show $|E_{n+1}-E_n|$ as a function of (n-N) for the case that $\theta = 0$ and that

$c_I = 1$ m/sec

$K = 10^{-2}$ m^{-1}

$N = 250 \cos\theta$

$k = 2.5 \cos^2\theta$

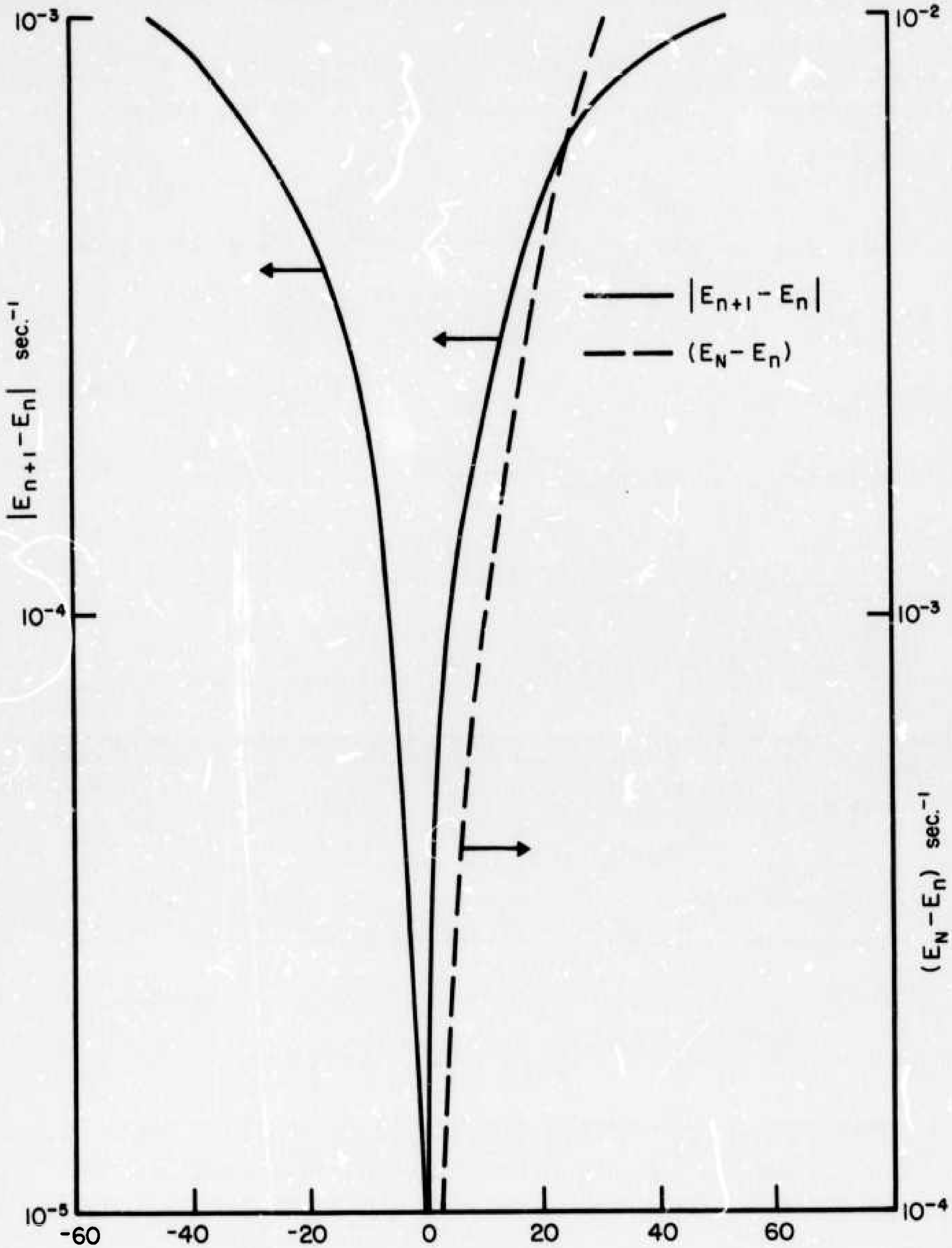

Figure 1. The difference in frequency between adjacent modes is given by the solid curve. The dashed curve shows the frequency shift from resonance.

The quantity $(E_N - E_n)$ is also shown in Fig. (1) for these same parameters.

To discuss the resonant strong coupling ($S >> 1$) solution, we consider mode numbers near $n = M \approx N$ and define an integer ν_o as most closely approximating the relation

$$\nu_o = \sqrt{S} \quad . \tag{3.31}$$

For $|n-M| \leq \nu_o$ we set $E_n = E_M$ and

$$V_{n,n\pm 1} \cong V_o = U_o \, KM/2 \tag{3.22}$$

in Eq. (3.18), which becomes simply

$$\varepsilon_\lambda \, \psi_\lambda(n) = \tfrac{1}{2}\left[\psi_\lambda(n+1) + \psi_\lambda(n-1)\right] \quad , \tag{3.33}$$

where,

$$\varepsilon_\lambda \equiv (E^\lambda - E_M)/(2V_o) \quad . \tag{3.34}$$

The boundary conditcns (3.19) are now taken as

$$\psi_\lambda(M \pm \nu_o) = 0 \quad . \tag{3.35}$$

This choice of boundary condition will not limit the validity of our results for initial modes having n-values well within

the interval $M-\nu_o < n < M+\nu_o$ and times short enough that modes are not excited near $n = (N\pm\nu_o)$ or $(M\pm\nu_o)$.

The eigenfunctions of Eqs. (3.33)-(3.35) are then

$$\psi_\lambda(n) = \nu_o^{-\frac{1}{2}} \sin\left[\frac{\pi}{2\nu_o} \lambda (n - M + \nu_o)\right] ,$$

$$\varepsilon_\lambda = \cos\left(\frac{\pi}{2\nu_o} \lambda\right) ,$$

$$n = M - \nu_o + 1, \ldots, M + \nu_o - 1 ,$$

$$\lambda = 1, 2, \ldots 2\nu_o - 1 . \tag{3.36}$$

Equation (3.33) can be generalized if we use (3.16) in (3.18), but continue to set $E_n = E_M$. The resulting equation is

$$(E^\lambda - E_M)\, \psi_\lambda(n) = \left(\frac{U_o K}{2}\right) \sqrt{n} \left[\sqrt{n+1}\, \psi_\lambda(n+1) + \sqrt{n-1}\, \psi_\lambda(n-1)\right] .$$

If we define

$$\phi_\lambda(s) = \sum_n s^{n-1} \sqrt{n}\, \psi_\lambda(n) , \tag{3.37}$$

there results the equation

$$v_\lambda\, \phi_\lambda = \frac{d}{ds}\left[(1+s^2)\, \phi_\lambda\right] , \tag{3.38}$$

where

$$v_\lambda \equiv 2(E^\lambda - E_M)/(U_o K) \quad . \tag{3.39}$$

On integrating (3.38), we obtain

$$\phi_\lambda(s) = \frac{C_\lambda}{1+s^2} \exp\left[v_\lambda \tan^{-1}(s)\right] \quad , \tag{3.40}$$

where C_λ is a constant.

Following Rosenbluth[7], we can give a somewhat more elaborate analysis near resonance by treating n as a continuous variable and using the approximation (3.24) for E_n. Using the approximation (3.22), we set M = N and obtain

$$\left[E^\lambda - E_N - 2V_o + \alpha(n-N)^2\right] \psi_\lambda(n) \cong V_o \frac{d^2\psi_\lambda(n)}{dn^2} \quad . \tag{3.41}$$

We now have $V_o = U_o KN/2$, since we have taken M = N. In deriving Eq. (3.41) we have neglected a term of order $\left(\frac{d\psi_\lambda}{dn} / N\right)$ compared with $\psi_\lambda(n)$ and have neglected derivatives of higher order than the second.

Equation (3.41) is the "harmonic oscillator" equation with "spring constant" α. For $\cos\theta > \left(\frac{2}{3}\right)^{\frac{1}{2}}$ [Eq. (3.24)] α is positive and the "restoring force" limits the spreading of modes. For $\cos\theta < \left(\frac{2}{3}\right)^{\frac{1}{2}}$ the force is "anti-restoring" and indefinite spreading in mode-space occurs.

If we define the quantities,

$$v \equiv (n-N)/S^{\frac{1}{4}} \quad ,$$

$$\Delta_\lambda \equiv \left[2 - (E^\lambda - E_N)(V_o)^{-1}\right] / S^{\frac{1}{2}} \tag{3.42}$$

and $\Psi_\lambda(v) \equiv \psi_\lambda(n)$

then Eq. (3.41) can be written as

$$\frac{d^2\Psi_\lambda}{dv^2} + (\Delta_\lambda - v^2)\Psi_\lambda = 0 \quad . \tag{3.43}$$

This has the solutions,

$$\Delta_\lambda = 2\lambda + 1 \; , \qquad \lambda = 0, 1, 2, \ldots$$

$$\Psi_\lambda(v) = \left[S^{\frac{1}{4}}\, 2^\lambda\, \lambda!\, \sqrt{\pi}\right]^{-\frac{1}{2}} H_\lambda(v) \exp\left[-v^2/2\right] \tag{3.44}$$

$$= \psi_\lambda(n)$$

where H_λ is the Hermite polynomial

$$H_\lambda(v) = \frac{\lambda!}{2\pi i} \oint e^{2vz-z^2} \frac{dz}{z^{\lambda+1}} \tag{3.45}$$

and the contour of integration is a circle about $z = 0$[8].

The normalization of Eq. (3.44) is so chosen that

$$\int_{-\infty}^{\infty} dn \, \psi_\lambda(n) \, \psi_{\lambda'}(n) = \delta_{\lambda\lambda'}$$

and the eigenvalues of E^λ are of the form

$$E^\lambda = E_N + V_o \left(2 - S^{-\frac{1}{2}}\right) - \lambda\delta\omega \tag{3.46}$$

where

$$\delta\omega \equiv 2V_o \, S^{-\frac{1}{2}} = \left[\frac{C_I U_o}{2} \left(3 \cos^2\theta - 2\right)\right]^{\frac{1}{2}} K \quad . \tag{3.47}$$

The above result is similar to that obtained by Rosenbluth[7] for the one dimensional impulsively applied internal wave; the eigenvalues being identical for $\theta = 0$.

4.0 The Initial Value Condition

The general form of $b_{\underset{\sim}{k}}^{(+)}$, Eq. (3.5) is seen to be

$$b_{\underset{\sim}{k}}^{(+)} = e^{-in\Omega t} \gamma_n \psi(n,t) , \tag{4.1}$$

where $\psi(n,t)$ is a solution to Eq. (3.16). We suppose that at time $t = 0$,

$$\psi(n,o) = \delta_{n,M} \frac{Q^{(M)}}{\sqrt{2}\, \gamma_M} , \tag{4.2}$$

corresponding to an initial simple wave of mode number M. For other times we write

$$\psi(n,t) \equiv e^{-iE_M t} \frac{Q^{(M)}}{\sqrt{2}\, \gamma_M} \Phi(n,t) . \tag{4.3}$$

The complex amplitude Z $\left[\text{Eq. (2.9)}\right]$ is then

$$Z = \frac{e^{i(k_M x - \omega_M t)}}{k_M} Q^{(M)} \sum_n \frac{\gamma_n\, k_M}{k\, \gamma_M} e^{i(n-M)K\xi} \Phi(n,t), \tag{4.4}$$

with ξ given by Eq. (2.2) and $k_M \equiv \left[M^2 K^2 + p^2\right]^{\frac{1}{2}}$.

To first order in M^{-1},

$$\frac{k_M\, \gamma_n}{k\, \gamma_M} = 1 + \frac{\cos^2\theta}{4M} (n-M) ,$$

so (4.4) becomes now $\cos\theta = MK/k_M$

$$Z = \frac{e^{i(k_M x - \omega_M t)}}{k_M} Q^{(M)} G^{(M)} \quad ,$$

$$G^{(M)} = \left[1 - i \frac{\cos^2\theta}{4MK} \frac{\partial}{\partial x}\right]\left\{\sum_n e^{i(n-M)K\xi} \phi(n,t)\right\} \quad . \qquad (4.5)$$

If we expand in terms of the eigenfunctions of Eq. (3.18) and use (4.2), we obtain

$$\psi(n,t) = \frac{Q^{(M)}}{\sqrt{2}\,\gamma_M}\left[\sum_\lambda e^{-iE^\lambda t} \psi_\lambda(n)\, \psi_\lambda(M)\right] \quad . \qquad (4.6)$$

Evaluation of (4.5) when we can use the perturbation solution (3.28) is straightforward. There results

$$G^{(M)} \cong 1 - \left(\frac{U_o KM}{2}\right)\left(1 + \frac{1}{2M}\left(1 + \frac{1}{2}\cos^2\theta\right)\right) e^{iK\xi}$$

$$\times \left[1 - e^{-i(E_{M+1}-E_M)t}\right] \Big/ (E_{M+1}-E_M)$$

$$- \left(\frac{U_o KM}{2}\right)\left(1 - \frac{1}{2M}\left(1 + \frac{1}{2}\cos^2\theta\right)\right) e^{-iK\xi}$$

$$\times \left[1 - e^{-i(E_{M-1}-E_M)t}\right] \Big/ (E_{M-1}-E_M) \quad . \qquad (4.7)$$

On expanding Eq. (4.7) in t near t = 0, we obtain

$$|G^{(M)}| \cong R_e \left(G^{(M)}\right)$$

$$\cong 1 + \left(\frac{U_o KMt}{2}\right)\left[(1 + \frac{1}{2}\cos^2\theta)/M\right]\sin(K\xi)$$

$$- \left(\frac{U_o KMt^2}{4}\right)\frac{\partial^2 E_M}{\partial M^2}\cos(K\xi) \quad . \qquad (4.8)$$

For $\theta = 0$ and $M = N$, this agrees with the results of Zachariasen[3] when an error in his work is corrected.

To obtain a more accurate description near resonance, we write $(K + V)$ in Eq. (3.16) in the form

$$K + V = H_o + H^1 ,$$

$$H_o\psi = E_M\,\psi(n) + V_o\left[\psi(n+1) + \psi(n-1)\right] ,$$

$$H^1\psi = (E_n - E_M)\,\psi(n) + V^1_{n,n+1}\,\psi(n+1) + V^1_{n,n-1}\,\psi(n-1) , \qquad (4.9)$$

where

$$V^1_{n,n\pm1} \cong V_o\,(n-M\pm\tfrac{1}{2})/N . \qquad (4.10)$$

We next take

$$\psi(n) = \psi^o(n) + \psi^1(n) , \qquad (4.11)$$

treat H^1 and ψ^1 as small, and specify that

$$i \frac{\partial \psi^o}{\partial t} = H_o \psi^o . \tag{4.12}$$

Substitution of (4.11) into (3.15) then gives the first order equation

$$i \frac{\partial \psi^1}{\partial t} - H_o \psi^1 \cong H^1 \psi^o \tag{4.13}$$

to determine ψ^1. If we set $\psi^1(n,t) = 0$ at $t = 0$, the boundary condition (4.2) gives us

$$\psi^o = \sum_\lambda e^{-iE^\lambda t} \psi_\lambda(n) \psi_\lambda(M) \frac{Q^{(M)}}{\sqrt{2}\, \gamma_M} , \tag{4.14}$$

where E^λ and the ψ_λ are given by Eqs. (3.34) and (3.36).

The transformation (4.3) then lets us write

$$\begin{aligned} \phi^o(n) &= \sum_\lambda e^{-i2V_o t \varepsilon_\lambda} \psi_\lambda(n) \psi_\lambda(M) \\ &\cong \frac{1}{\pi} \int_o^\pi d\alpha\, e^{-i(2V_o t)\cos\alpha} \cos(\nu\alpha) \\ &= i^\nu J_\nu(-2V_o t) . \end{aligned} \tag{4.15}$$

Here we have set $\alpha = \frac{\pi}{2\nu_o} \lambda$, replaced the sum by an integral, and have defined

$$\nu \equiv (n-M). \tag{4.16}$$

In anticipation of the evaluation of (4.5), we use (4.15) and find that

$$\sum_{\nu} e^{i\nu K\xi}\,\phi^{o}(n) = \exp\left\{-iU_o KMt\,\cos(K\xi)\right\} . \tag{4.17}$$

If we neglect the term of O(1/N) and substitute the above into (4.5) we see that the effect of the surface current is to modulate the phase velocity of the surface wave. This velocity is

$$C_M = \frac{\omega_M}{k_M} + U_o\cos\theta\,\cos(K\xi) , \tag{4.18}$$

a result that could have been deduced from elementary considerations.

To obtain corrections of O(1/N) we must integrate Eq. (4.13). On setting

$$\psi^{1}(n) \equiv e^{-iE_M t}\,\frac{Q^{(M)}}{\sqrt{2}\,\gamma_M}\,\phi^{1}(n) , \tag{4.19}$$

we obtain

$$\phi^{1}(M\pm 1) \cong \left\{\mp it/(2N) + (E_{M\pm 1}-E_M)\,\frac{t^2}{2}\right\}V_o \tag{4.20}$$

for small t. Evaluation of (4.5) finally gives us

$$G^{(M)} \cong \left[1 + \frac{V_o t}{N} \frac{1}{2} \cos^2\theta \sin(K\xi)\right] e^{-i2V_o t \cos(K\xi)}$$

$$+ V_o \left\{\frac{t}{N} \sin(K\xi) - \frac{t^2}{2} \frac{\partial^2 E_M}{\partial M^2} \cos(K\xi)\right\} . \qquad (4.21)$$

This agrees with the Zachariasen expression (4.8) for short times. Since the second term above has been evaluated only for small t, we cannot use (4.21) for late times. From Eq. (4.15) we see that the probability that mode number n is excited is proportional to $\left[J_\nu (2V_o t)\right]^2$. Thus, when

$$t \gtrsim t_p \equiv \frac{\pi}{4} \frac{\nu_o}{V_o} ,$$

the neglect of $(E_n - E_M)$ in Eq. (3.33) is no longer valid and we must use the Rosenbluth equation (3.41) to study the interaction.

Before doing this, it is instructive to construct the wave amplitude Z using the function ϕ_λ of Eq. (3.40). After some simplification, and with the identification that

$$s = e^{iK\xi} ,$$

we obtain

$$Z = e^{-iE_M t} (\sqrt{2}\, \gamma_M / k_M)\, e^{i(py+K\xi)} \left[1 + \left((\cos^2\theta - 2)/(4M)\right)\right.$$

$$\left.\left(s \frac{d}{ds} + 1 - M\right)\right] \Gamma(s,t), \qquad (4.22)$$

where

$$\Gamma(s,t) = \left[\sum_{\lambda} e^{-iE^{\lambda}t} \phi_{\lambda}(s)\right] e^{iE_M t} .$$

$$= \int \frac{dv\ C(v)}{1+s^2} e^{v(\theta-\alpha)} . \tag{4.23}$$

Here we have replaced the sum over λ by an integral over

$$v \equiv 2(E^{\lambda}-E_M)/(U_o K) \tag{4.24}$$

and have set

$$\theta \equiv \tan^{-1}(s) , \quad \alpha \equiv itU_o K/2 . \tag{4.25}$$

The coefficient $C_{\lambda} \equiv C(v)$ is chosen to satisfy the boundary condition that

$$\Gamma(s,o) = s^{M-1} Q(m)/(\sqrt{2}\,\gamma_M) . \tag{4.26}$$

Using (4.26) we obtain

$$\Gamma(s,t) = \tan^{M-1} (\theta-\alpha) \frac{[1 + \tan^2(\theta-\alpha)]}{(1 + \tan^2\theta)} \frac{Q^{(M)}}{\sqrt{2}\,\gamma_M} \tag{4.27}$$

On evaluating (4.22) we obtain the envelope function:

$$|G^{(M)}| = (1-\sigma^2)\left[1 + \sigma\left(\frac{1}{2}\cos^2\theta - 1\right)\sin(K\xi)/\right.$$

$$\left.\left(1 + \sigma^2 - 2\sigma\sin(K\xi)\right)\right] / \left(1 + \sigma^2 - 2\sigma\sin(K\xi)\right) \quad . \tag{4.28}$$

Here

$$\sigma \equiv \tanh\,(U_o Kt/2). \tag{4.29}$$

For small t, $\sigma = V_o t/N$ and Eq. (4.28) is in agreement with the linear t-dependent terms of Eq. (4.21). Eq. (4.28) is singular for $\sigma = 1$, $\sin K\xi = 1$. Including the effect of the $(E_n - E_M)$ term in (3.18) would remove this singularity

For $\sigma << 1$ we can simplify Eq. (4.27) to obtain

$$G^{(M)} \cong \left[1 + \frac{V_o t}{N}(1 + \tfrac{1}{2}\cos^2\theta)\,\sin(K\xi)\right] \exp\left[-2iV_o t\,\cos(K\xi)\right] \quad , \tag{4.29}$$

which modifies Eq. (4.21) with a plausible phase factor correction and, of course, omits the term involving $\partial^2 E_M/\partial M^2$.

Returning to the Rosenbluth equation (3.41), the general form of $\psi(n,t)$ is

$$\psi(n,t) = \sum_\lambda e^{i\delta\omega\lambda t}\, C_\lambda\, \psi_\lambda(n) \quad , \tag{4.30}$$

where we have dropped a phase factor,

$$\exp\left\{-i\left[E_N + 2V_o\left(1 - \frac{1}{\sqrt{S}}\right)\right] t\right\} \quad ,$$

and $\delta\omega$ and $\psi_\lambda(n)$ are given by Eqs. (3.42) to (3.47). The

coefficients C_λ are

$$C_\lambda = \int dn\, \psi_\lambda(n)\, \psi(n,o)\ . \tag{4.31}$$

For an initial Gaussian,

$$\psi(n,o) = \exp\left[-\ (v-v_o)^2/2A\right] \frac{Q^{(M)}}{\sqrt{2}\ \gamma_M} \tag{4.32}$$

where $v_o = (M-N)/S^{\frac{1}{4}}$ and A is a parameter, (4.31) is easily evaluated and we find

$$\psi(n,t) = \exp\left\{v_o^2\ \left[1/(1+A') - 1/(1+A)\right]/2\right\}$$

$$\exp\left\{-(v-v_o')^2/(2A')\right\}\ \frac{Q^{(M)}}{\sqrt{2}\ \gamma_M}\ . \tag{4.33}$$

Here

$$A' = \frac{\left[1 - \frac{1-A}{1+A}\ e^{2i\delta\omega t}\right]}{\left[1 + \frac{1-A}{1+A}\ e^{2i\delta\omega t}\right]}\ ,$$

$$v_o' = v_o\left[\frac{1-A'^2}{1-A^2}\right]^{\frac{1}{2}}\ . \tag{4.34}$$

A special case of some interest is that of the "non-spreading" wave packet, with

$$A = 1,\ \ A' = 1\ ,$$

$$v_o' = v_o\ e^{2i\delta\omega t}\ . \tag{4.35}$$

In this case

$$|\psi(n,t)|^2 = \left|\frac{Q^{(M)}}{\sqrt{2}\ \gamma_M}\right|^2 \exp\left\{-\left[n - N - (M-N)\cos(\delta\omega t)\right]^2/S^{\frac{1}{2}}\right\}. \tag{4.36}$$

The most probable value of n is

$$n = N + (M-N)\cos(\delta\omega t) \tag{4.37}$$

Returning to Eq. (4.33), we re-write this as

$$\psi(n,o) \sim \exp\left[-\ (n-M)^2/(2AS^{\frac{1}{2}})\right] \quad .$$

To satisfy our boundary condition (4.2) we evidently require that

$$2AS^{\frac{1}{2}} << 1 \ . \tag{4.38}$$

Reference to Eq. (4.34) shows then that when

$$e^{2i\delta\omega t} = -1$$

$|A'|$ is very large and we encounter Rosenbluth's "pile-up" near $\cos K\xi = 1$. This "pile-up" is greatest at

$$t = \tau_p \equiv \frac{\pi}{2\delta\omega} \quad . \tag{4.39}$$

Rosenbluth's estimate gives

$$\begin{aligned} G^{(M)} &\cong 1 + R^{(M)}, \qquad R^{(M)} > 0 \\ &= 1, \qquad\qquad\quad R^{(M)} < 0 \end{aligned} \tag{4.40}$$

$$R^{(M)} \equiv \frac{\sin\left[2.8\ S^{1/8}\ \sin(K\xi)\right]}{\sin(K\xi)} - 1 \quad . \tag{4.41}$$

On setting M = N in Eq. (4.9), we model $G^{(M)}$ as follows:

$$\begin{aligned} G^{(M)} \cong \Bigg\{ 1 &+ \left(\frac{t}{\tau_v}\right) \frac{(1 + \frac{1}{2}\cos^2\theta)}{N} \sin(K\xi) \\ &+ \frac{\cos(K\xi)}{S} \left(\frac{t}{\tau_v}\right)^2 \left(1 - \frac{t}{\tau_p}\right)^2 \\ &+ \left(\frac{t}{\tau_p}\right)^4 R^{(M)} \Bigg\} \exp\left[-iMKU(\xi)t\right] , \quad t \leq \tau_p \end{aligned} \tag{4.42}$$

where

$$\tau_v = (U_o KN/2)^{-1} , \tag{4.43}$$

and $\tau_p = \frac{\pi}{4} \tau_v S^{1/2}$, an alternate form for Eq. (4.39).

For applications it will be desirable to re-label the envelope function (4.5) with the wave number $k = (MK,p)$,

setting

$$G_{\underset{\sim}{k}} \equiv G^{(M)} . \tag{4.44}$$

For times much larger than τ_p , phase mixing substantially reduces the resonant modulation of the envelope function. We note that for $t = \tau_p$, the variable (4.29) is

$$\sigma = \tanh\left(\frac{\pi}{4} S^{\frac{1}{2}}/N\right) ,$$

which we anticipate to be small compared with unity. This seems to justify the linear approximation in (4.42).

5.0 The WKB Approximation

For a mode number M sufficiently far from resonance that we can set

$$E_{M+1} - E_M \cong \frac{\partial E_M}{\partial M} \quad ,$$

the parameter (3.29) is

$$S_M = \left| \frac{U_o M}{2(c_I - c_g \cos\theta)} \right| \quad . \tag{5.1}$$

If this is greater than unity the perturbation solution (3.28) cannot be used, but the WKB approximation can be. To develop this we re-write Eq. (2.3) in terms of t, $\xi = x - c_I t$, and y:

$$\frac{\partial \phi}{\partial t} + V \frac{\partial \phi}{\partial \xi} + g\zeta = 0 \quad ,$$

$$\frac{\partial \zeta}{\partial t} + V \frac{\partial \zeta}{\partial \xi} + \zeta \frac{dU}{d\xi} = \textcircled{\#}_s \, \phi \tag{5.2}$$

Here

$$V \equiv U - c_I \quad . \tag{5.3}$$

Next, we write

$$\phi = A(\xi)\, e^{i(py-\omega t)}\, e^{i\int^{\xi} q(\xi')d\xi'}$$

$$\zeta = a(\xi)\, e^{i(py-\omega t)}\, e^{i\int^{\xi} q(\xi')d\xi'} \quad . \tag{5.4}$$

As usual, we consider q, A and a to be slowly varying functions and neglect higher than first derivatives of these as well as products of derivatives.

Treating A' and q' as small, we then work out the relation[9]

$$\oplus_s\phi \cong \left\{kA - i\,\frac{q}{k}\,A' - \frac{i}{2}\left[1 - \left(\frac{q}{k}\right)^2\right]q'A\right\} e^{i(py-\omega t)}\, e^{i\int^{\xi} qd\xi'} , \tag{5.5}$$

where

$$k \equiv (p^2 + q^2)^{\frac{1}{2}} \quad .$$

Substitution of Eqs. (5.5) and (5.4) into Eq. (5.2) is straightforward and A can be eliminated from the coupled equations. There results $\left[\text{here } a' \equiv \frac{da}{d\xi}\text{ , etc., and } c_g \equiv \frac{d\omega_k}{dk}\right]$

$$\left[(\omega-qV)^2 - \omega_k^2\right]a \quad + ik\left\{\left[4\, c_g(V + c_g\, \cos\theta)\right]\, a'\right.$$

$$\left. + a\left[2\, c_g\, V' + 2\, c_g'\, V + 4\, c_g\, c_g'\, \cos\theta + 2\, c_g^2\, \sin^2\theta\frac{q'}{k}\right]\right\} = 0 \tag{5.6}$$

The first term, involving no derivatives, gives the usual dispersion relation

$$\omega - q\,V = \omega_k \equiv \sqrt{gk} \tag{5.7}$$

for waves traveling in the positive x-direction. The terms involving derivatives may be integrated to yield

$$\sqrt{c_g(V + c_g \cos\theta)}\; a(\xi) = \text{const.} \tag{5.8}$$

To first order in U, Eq. (5.8) becomes

$$a(\xi) \cong a_o\left\{1 + \frac{U}{4}\cos\theta\left[c_I - 2\,c_g\cos\theta + 2\,c_g\sin^2\theta\right]/(c_I - c_g\cos\theta)^2\right\}$$

$$\equiv a_o\,G^{(M)}\ , \tag{5.9}$$

where again M labels the mode.

For sufficiently short wavelengths, we may neglect c_g compared to c_I above. Then

$$a(\xi) \cong a_o\left[1 + \frac{U\cos\theta}{4\,c_I}\right] \quad . \tag{5.10}$$

The short wavelength will be subject to modulation effect due to interaction with long wavelength waves. Let $\underset{\sim}{\ell}$ be

a wave number for which $\ell << k$. In the linear approximation we have

$$\hat{U}_{\underset{\sim}{\ell}} = V_\ell\, q_\ell \cos(\underset{\sim}{\ell}\cdot\underset{\sim}{x}-\omega_\ell t)$$

$$= V_\ell\, Q_{\underset{\sim}{\ell}}\, G_{\underset{\sim}{\ell}} \cos(\underset{\sim}{\ell}\cdot\underset{\sim}{x}-\omega_\ell t) \quad . \tag{5.11}$$

Here $V_\ell = \omega_\ell/\ell$ and $G_{\underset{\sim}{\ell}}$ is the envelope modulation function for the wave ℓ. Then, in Eq. (5.8) we have

$$V = U_\ell\,(G_{\underset{\sim}{\ell}}-1) - (V_\ell - U_\ell) \tag{5.12}$$

where

$$U_\ell = V_\ell\, Q_{\underset{\sim}{\ell}} \cos(\underset{\sim}{\ell}\cdot\underset{\sim}{x}-\omega_\ell t) \quad . \tag{5.13}$$

Linearizing the relation (5.8) in $(G_{\underset{\sim}{\ell}}-1)$, and again assuming $c_g = \frac{d\omega_k}{dk} << V_\ell$, we find

$$a(\xi) \cong a_o \left\{ 1 + \frac{\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{\ell}}}{4}\, \frac{Q_{\underset{\sim}{\ell}} \cos(\underset{\sim}{\ell}\cdot\underset{\sim}{x}-\omega_\ell t)}{1 - Q_{\underset{\sim}{\ell}} \cos(\underset{\sim}{\ell}\cdot\underset{\sim}{x}-\omega_\ell t)}\,(G_{\underset{\sim}{\ell}}-1) + \frac{U}{4}\frac{\cos\theta}{c_I} \right\} \quad . \tag{5.14}$$

Here we have added the direct interaction given by Eq. (5.10).

On squaring and performing an ensemble average, we obtain

$$\langle |a^2(\xi)|\rangle \cong \langle |a_o|^2\rangle \left\{1 + \frac{\underset{\sim}{k}\cdot\hat{\underset{\sim}{\ell}}}{4}\; Q^2_{\underset{\sim}{\ell}}\; \mathrm{Re}(G_{\underset{\sim}{\ell}}-1) + \frac{\cos\theta}{4}\,\frac{U}{c_I}\right\} \; .$$

Here we have neglected higher moments than the second of the slope function $q_{\underset{\sim}{\ell}}$. If we sum over ℓ and introduce the spectral function (2.12), this becomes

$$\langle |a^2(\xi)|\rangle \equiv \langle |a_o^2|\rangle \; |G_{\underset{\sim}{k}}|^2$$

$$= \langle |a_o|^2\rangle \left\{1 + \sum_{\ell} \frac{\ell^2}{2}\, (\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{\ell}})\; P(\underset{\sim}{\ell})\; \mathrm{Re}(G_{\underset{\sim}{\ell}}-1) + \frac{\cos\theta}{4}\,\frac{U}{c_I}\right\} \; . \qquad (5.15)$$

References

1. J.A. Thomson, K.M. Watson, B.J. West, " A Mode Coupling Description of Ocean Wave Dynamics: Part I" PD 73-030.

2. K.M. Watson, B.J. West, B.I. Cohen, "Energy Spectra of the Ocean Surface, Part II: Interaction with Surface Current", PD 73-037.

3. F. Zachariasen, "Internal Wave-Surface Wave Interaction Revisited", IDA-Jason Paper P-853, March 1972.

4. D.M. Milder, unpublished notes, RDA, December 1972.

5. A derivation of Eqs.(2.3) is given by B. Cohen, K. Watson, and B. West in Part IV, "Coupling of Surface and Internal Gravity Waves: A Hamiltonian Model", PD 73-032.

6. O.M. Phillips, "The Dynamics of the Upper Ocean".

7. M. Rosenbluth, "Surface Waves in the Presence of an Internal Wave", Jason Paper P-832, Institute for Defense Analyses, Nov. 1971.

8. See, for example, M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards Appl. Math. Series 55, U.S. Government Printing Office.

9. The relation (5.5) is easily verified on operating once again with $\textcircled{\#}_s$ on both sides of (5.5).

MISSION
of
Rome Air Development Center

RADC is the principal AFSC organization charged with planning and executing the USAF exploratory and advanced development programs for electromagnetic intelligence techniques, reliability and compatibility techniques for electronic systems, electromagnetic transmission and reception, ground based surveillance, ground communications, information displays and information processing. This Center provides technical or management assistance in support of studies, analyses, development planning activities, acquisition, test, evaluation, modification, and operation of aerospace systems and related equipment.